Equation (5) may be a near optimum compromise between accuracy and simplicity with the estimation of damping ζ_m the critical factor in most applications.[10-12]

There are other effects which could be included in a more refined analysis. For example, the motion of the panel may change the near field turbulent-flow pressure fluctuations which excite the panel. However, the effect is likely to be small under most circumstances and, in any event, our present meager knowledge does not justify its inclusion in a simplified analysis.

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Practical Aspect of the Generalized Inverse of a Matrix

Hermann J. Hassig*

Lockheed-California Company, Burbank, Calif.

THE rules of matrix algebra make it possible to write m equations with n unknowns x in the form of a matrix equation

$$[A]_{m,n}\{x\}_{n,1}=\{y\}_{m,1} \tag{1}$$

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*Research and Development Engineer, Advanced Design and Technologies.

In the following discussion it is assumed that the equations represented by Eq. (1) are linearly independent. If $m=n$ there is a unique solution for $\{x\}$ in terms of the right-hand side $\{y\}$

$$\{x\}=[A]^{-1}\{y\} \tag{2}$$

The solution for $\{x\}$ is not affected by multiplication of any equation in Eq. (1) by a finite nonzero number.

If $m>n$ there are more equations than unknowns: the problem is over determined. In the so-called least-squares approach, a unique solution can be found by minimizing

$$\sum_{i=1}^{m}\epsilon_i^2$$

where ϵ is an "error," the difference between the left-hand and right-hand sides of each individual equation in Eq. (1)

$$\{\epsilon\}_{m,1}=[A]_{m,n}\{x\}_{n,1}-\{y\}_{m,1} \tag{3}$$

The solution is given by

$$\{x\}_{n,1}=\left[[A^T]_{n,m}[A]_{m,n}\right]^{-1}[A^T]_{n,m}\{y\}_{m,1} \tag{4}$$

where $[A^T]$ is the transpose of $[A]$.

For a given matrix $[A]$, Eq. (4) defines a unique solution for $\{x\}$. The solution, however, is affected by multiplying one or more of the equations in Eq. (1) by arbitrary numbers.

The preceding formulations are in general use and well understood. The purpose of this Note is to add to the understanding of the case that $m<n$. That is the case with fewer equations than unknowns: the problem is underdetermined.

From the general theory of equations it is known that if $m<n$, Eq. (1), does not lead to a unique solution $\{x\}$. In fact, an infinite number of solutions $\{x\}$ are possible. To make it possible to define a unique solution $\{x\}$ $(n-m)$ equations must be added to Eq. (1), or, equivalently, some constraints must be put on the relation between the elements of $\{x\}$.

In the literature, a generalized inverse of $[A]_{m,n}$ $(m<n)$ is given the notation $[A]^+_{n,m}$. Parallelling Eq. (2) it defines $\{x\}$ in terms of $\{y\}$ as follows

$$\{x\}_{n,1}=[A]^+_{n,m}\{y\}_{m,1} \tag{5}$$

An expression for $[A]^+$ can be found in many places in the literature.[1-8] This author, however, has failed to find a specific reference to the additional constraints on the elements of $\{x\}$ that make it possible to determine a unique matrix $[A]^+$.

In the following derivation it is shown that an expression for $[A]^+$ identical to the one found in the literature is found if the following constraint on $\{x\}$ is applied

$$\{x\}_{n,1}=\alpha_1\{a_1\}_{n,1}+\alpha_2\{a_2\}_{n,1}+\ldots\alpha_m\{a_m\}_{n,1} \tag{6}$$

where $\{a_1\}$, $\{a_2\}$,...$\{a_m\}$ are the transposes of the m rows of $[A]_{m,n}$. Equation (6) can be written as

$$\{x\}_{n,1}=[A^T]_{n,m}\{\alpha\}_{m,1} \tag{7}$$

Substituting Eq. (7) into Eq. (1) leads to

$$[A]_{m,n}[A^T]_{n,m}\{\alpha\}_{m,1}=\{y\}_{m,1} \tag{8}$$

Thus, Eq. (1) with n unknowns x has been transformed into Eq. (8) which contains m equations with m unknowns α.

Solving Eq. (8) for $\{\alpha\}$ gives

$$\{\alpha\}_{m,1}=\left[[A]_{m,n}[A^T]_{n,m}\right]^{-1}\{y\}_{m,1} \tag{9}$$

With Eq. (7) the solution for $\{x\}$ is

$$\{x\}_{n,1} = [A^T]_{n,m}\left[[A]_{m,n}[A^T]_{n,m}\right]^{-1}\{y\}_{m,1} \qquad (10)$$

Thus, the generalized inverse of $[A]_{m,n}$ for $m<n$ can be defined as

$$[A]^+_{n,m} = [A^T]_{n,m}\left[[A]_{m,n}[A^T]_{n,m}\right]^{-1} \qquad (11)$$

It should be noted that

$$[A]_{m,n}\ [A]^+_{n,m} = [1]$$

However,

$$[A]^+_{n,m}\ [A]_{m,n} \neq [1]$$

Obviously Eq. (4) can also be considered as defining a generalized inverse. Thus the generalized inverse of $[A]_{m,n}$ for $m>n$ can be defined as

$$[A]^+_{n,m} = \left[[A^T]_{n,m}[A]_{m,n}\right]^{-1}[A^T]_{n,m} \qquad (12)$$

Thus, the expression for $[A]^+_{n,m}$ depends on the relative magnitude of m and n. If $m=n$ both Eqs. (11) and (12) reduce to $[A]^+ = [A]^{-1}$.

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Mach Reflection Using Ray-Shock Theory

Brian Edward Milton*

University of New South Wales, Kensington, N.S.W., Australia

Introduction

THE Mach reflection of shock waves at plane corners, as shown in Fig. 1, can be described by the three-shock theory (von Neumann)[1] or the ray-shock theory (Whitham).[2] The former gives the strength and direction of the Mach stem and reflected wave relative to the triple-point locus, and it may be used to locate the locus if the stem is assumed to be straight and normal to the wall. This assumption is inherent in the ray-shock approach, which uses the average strength of the Mach stem to determine its area and hence the triple-point position. Both theories, however, show noticeable deviation from experimental measurements of the triple-point locus position over the range of corner angles at which Mach reflection occurs. For simple corners, the three-shock theory is appropriate. However, for calculations of the shock front strength during repeated reflections in converging cavities of finite angle change or with curved walls, the ray-shock theory is more convenient and, in the latter case, is the only one to apply. Repeated applications of the theory to multiple reflections magnify errors occurring due to mislocation of the triple-point locus and an improvement in accuracy is therefore required. This has been achieved by modifying the area, Mach number (A, M) function used in the theory. Good agreement is now available in the range where the straight stem assumption conforms to reality.

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*Lecturer, School of Mechanical and Industrial Engineering.

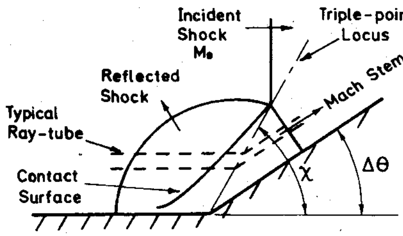

Fig. 1 Mach reflection configuration.

Area Change Function

The ray-shock theory consists of a set of equations based entirely on geometrical considerations which are related by the area change function first given by Chester.[3] This equation is usually written in the form

$$dA/A = -2M\, dM/K(M)\,(M^2-1) \qquad (1)$$

where $K(M)$ is a monotonic function of M ranging from 0.5 for weak shocks to 0.394 for strong shocks. Integration of Eq. (1) has been carried out by Chisnell[4] to give the motion of a shock which is unaffected by disturbances from behind. A recent paper by Yousaf[5] shows that a more valid form of Eq. (1) for imploding shocks which includes interaction terms λ is

$$(dA/A)(1+\lambda) = -2M\, dM/K(M)\,(M^2-1) \qquad (2)$$

His approach follows that of Whitham,[6] who derived Eq. (1) by examining the characteristics overtaking the shock. Alternative derivations are by Rosciszewski,[7] who integrated between neighboring characteristics, and Oshima et al,[8] who integrated along characteristics to obtain neglected correction terms.

Modified Area Change Function

Whitham's approach considers the flow equations in characteristic form

$$du + (dp/\rho a) + (ua/u+a)\,dA/A = 0 \quad \text{on } C^+ \qquad (3a)$$

$$du - (dp/\rho a) + (ua/u-a)\,dA/A = 0 \quad \text{on } C^- \qquad (3b)$$

Symbols p, ρ, a, and u are pressure, density, acoustic velocity, and velocity, respectively.

Substitution of the Rankine-Hugoniot conditions at the main shock front into the C^+ equation and subsequent manipulation give Eq. (1). This neglects interaction terms, as can be shown by an examination of the distance, time (x,t) plot of Fig. 2a. In the triangle $a\ b\ c$ formed by C^+ and C^- characteristics from point c away from the shock, which intersect the shock front line itself, terms on the C^- charac-